AIAA 81-1018R

Optimum Approximate-Factorization Schemes for Two-Dimensional Steady Potential Flows

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A procedure is described for constructing optimum approximate-factorization algorithms for solution of the two-dimensional equations for steady transonic potential flows. The factorizations are optimum in the sense that they should lead to the fastest convergence rates. It is shown that it is important to split the effect of transformation derivatives, normally appearing in the flow equations, correctly between the two factors in the basic factorization. An optimum value for the relaxation parameter is determined, and a procedure is described for constructing an optimum sequence of values for the acceleration parameter.

Introduction

FOR many years successive line overrelaxation (SLOR) has been the standard method for solving nonlinear equations in steady-flow problems. Although it is easy and reliable to use, this method's convergence rate is very slow, mainly because of the slowness with which information can be transmitted in the direction opposite to that in which the calculation proceeds, the transmission rate being one step length per iterative cycle (iteration). Several thousands of iterations are often needed to obtain a converged solution, although there are ways of reducing this to several hundreds of iterations, e.g., by adding a vortex flow in lifting problems to speed up transmission of information in the far field, or by performing a preliminary calculation on a coarse mesh.

Other solution methods have been developed more recently. Rapid elliptic solvers were used by Martin and Lomax[1] to calculate subcritical and mildly supercritical flows. After linearizing the flow equation by using the approximate solution obtained from the previous iteration, the resulting linear equation is solved exactly over the complete flowfield. The method converges in fewer iterations than other methods but has some limitations. Although it converges in a small number of iterations, the computation time for each iteration is greater than for other methods, and the method becomes unstable when a significant region of supercritical flow is present. Jameson[2] showed how to stabilize the method by combining it with SLOR, but this method is limited to fairly simple computational domains; in particular, coordinate stretching can be used in only one of the two directions.

Another fast method that is currently receiving attention is the multigrid method originally proposed by Federenko,[3] developed by Brandt,[4] and applied to transonic-flow problems by South and Brandt[5] and by Jameson.[6] In this method a calculation is performed over a range of successively coarser, then successively finer computational meshes. It is easy to see how information can be spread rapidly over the whole flowfield by use of the coarser meshes while the finer meshes are used for the local flow details. Put another way, if one imagines the error to be made up of Fourier components, then the coarser meshes are used to reduce the low-frequency errors while the finer meshes reduce the high-frequency errors. Another advantage of this method is that the computational time spent on the coarse meshes is very small. The main disadvantage is probably the organizational complexity. A solution method (e.g., SLOR or approximate factorization) needs to be developed first and then the multigrid structure superimposed.

The approximate-factorization (AF) method was developed for steady-transonic-flow calculations by Ballhaus, Jameson, and Albert,[7] who described two factorizations AF1 and AF2 (see below) that led to rapid convergence when applied to the transonic small perturbation (TSP) equation. Holst[8] applied a variation of AF2 to the full-potential equation. Baker[9] applied both AF1 and AF2 to the full-potential equation, using a circle plane mapping, and introduced a further factorization, AF3, that he found to be much faster than the other two, although the reason for this was not at the time evident. Other workers have had mixed success with AF schemes, sometimes finding particular schemes to be fast in particular solution domains and sometimes finding them disappointing, particularly on stretched Cartesian meshes, with optimal sets of acceleration parameters difficult to determine in general.[8,10] A sample convergence history for such a case, which will be discussed later, is shown in Fig. 5.

Before attempting to explain why the success of AF has been variable, the principle of the technique will be reviewed.

Approximate Factorization

The analysis in this paper will be restricted to solution of the difference problem

$$L(\phi) \equiv (A\overleftarrow{\delta X}\overrightarrow{\delta X} + B\overleftarrow{\delta Z}\overrightarrow{\delta Z})\phi = 0 \qquad (1)$$

where $\overrightarrow{\delta X}$ and $\overrightarrow{\delta Z}$ denote forward differencing (e.g., $\overrightarrow{\delta X}\phi_i \equiv \phi_{i+1} - \phi_i$) and $\overleftarrow{\delta X}$ and $\overleftarrow{\delta Z}$ denote backward differencing (e.g., $\overleftarrow{\delta X}\phi_i = \phi_i - \phi_{i-1}$). A and B are not assumed to be constants, so Eq. (1) can be taken to represent the discretized form of a flow equation (e.g., TSP or full-potential) with the lower-order and cross-derivative terms neglected. We consider here only subsonic flow, but the relevant upwind differences needed to deal with regions of supersonic flow can easily be included later. An iterative solution procedure to solve Eq. (1) can be written

$$N\Delta = \sigma L(\phi^n)$$

where Δ is the "correction" $\phi^{n+1} - \phi^n$ computed during the $n+1$th iteration and σ is a relaxation parameter.

In an AF scheme N is written as the product of two factors, for example the simplest form of the AF1 scheme of Ref. 7 is

$$(\alpha - A\overleftarrow{\delta X}\overrightarrow{\delta X})(\alpha - B\overleftarrow{\delta Z}\overrightarrow{\delta Z})\Delta = \alpha\sigma L(\phi^n)$$

Presented as Paper 81-1018 at the AIAA 5th Computational Fluid Dynamics Conference, Palo Alto, Calif., June 22-23, 1981; submitted July 6, 1981; revision received Dec. 1, 1981.

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where α is an acceleration parameter whose value needs careful selection in order to achieve rapid convergence.

Multiplying out the left-hand side of this expression gives

$$[\alpha^2-\alpha(A\overrightarrow{\delta}X\overleftarrow{\delta}X+B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)+AB\overrightarrow{\delta}X\overleftarrow{\delta}X\overrightarrow{\delta}Z\overleftarrow{\delta}Z]\Delta$$

$$=\alpha\sigma(A\overrightarrow{\delta}X\overleftarrow{\delta}X+B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)$$

if B is treated as locally constant. It can be seen that, with $\sigma=1$, we would be solving $L(\phi^{n+1})=0$ if we could neglect the first and third terms on the left-hand side. In practice it is found that by solving in turn the equations

$$(\alpha-A\overrightarrow{\delta}X\overleftarrow{\delta}X)F=\alpha\sigma L(\phi^n),\quad (\alpha-B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)\Delta=F$$

with a sequence of values of α, rapid convergence may be achieved. In Ref. 7 a Von Neumann stability analysis was performed and it was shown that, with $\sigma=2$, $A=1/\Delta X^2$, and $B=1/\Delta Z^2$ (ΔX, ΔZ are step lengths), the choice $\alpha=(\sin^2 \frac{1}{2}q\Delta Z)/\Delta Z^2$ leads to rapid convergence if a suitable range of values of q is taken.

Because the calculation sweeps in alternating directions, information is rapidly transmitted through the field, so that the method is potentially fast. It does, however, require careful choice of factors and a range of values for α.

In Ref. 7 it was suggested that AF1 factorization is unsuitable for supercritical flows, and an AF2 scheme,

$$(\alpha-A\Delta X\overrightarrow{\delta}X)(\alpha\overleftarrow{\delta}X/\Delta X-B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)\Delta=\alpha\sigma L(\phi^n)$$

was used successfully for flows with supersonic regions. For the TSP equation it is necessary to amend the first factor to $(\alpha-A\Delta X\overleftarrow{\delta}X)$ in regions of supersonic flow.

Baker's[9] AF3 scheme, in polar coordinates, is

$$(-\alpha A\Delta r\overrightarrow{\delta}r-B\overrightarrow{\delta}\theta\overleftarrow{\delta}\theta)(\alpha+\overleftarrow{\delta}r/\Delta r)\Delta=\alpha\sigma L(\phi^n)$$

which is similar to AF2 but with the factors reversed and A and B placed inside the first factor so that they need be calculated only during the first sweep.

Type of Factorization

Within the above format [i.e., matching the coefficient of α on the left-hand side to $L(\phi)$] there are basically only three possible factorizations:

$$\text{I:}\quad N=(\alpha A_1-A_2\overrightarrow{\delta}X\overleftarrow{\delta}X)(\alpha B_1-B_2\overrightarrow{\delta}Z\overleftarrow{\delta}Z)$$

$$\text{II:}\quad N=(\alpha A_1-A_2\overrightarrow{\delta}X)(\alpha B_1\overleftarrow{\delta}X-B_2\overrightarrow{\delta}Z\overleftarrow{\delta}Z)$$

$$\text{III:}\quad N=(\alpha A_1\overrightarrow{\delta}Z-A_2\overrightarrow{\delta}X)(\alpha B_1\overleftarrow{\delta}X-B_2\overleftarrow{\delta}Z)$$

where $A_2B_1=A$ and $A_1B_2=B$.

Writing $A_1B_1=1/\lambda$, these become:

$$\text{I:}\quad N=A_1(\alpha-\lambda A\overrightarrow{\delta}X\overleftarrow{\delta}X)(1/A_1\lambda)(\alpha-\lambda B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)$$

$$\text{II:}\quad N=A_1(\alpha-\lambda A\overrightarrow{\delta}X)(1/A_1\lambda)(\alpha\overleftarrow{\delta}X-\lambda B\overrightarrow{\delta}Z\overleftarrow{\delta}Z)$$

$$\text{III:}\quad N=A_1(\alpha\overrightarrow{\delta}Z-\lambda A\overrightarrow{\delta}X)(1/A_1\lambda)(\alpha\overleftarrow{\delta}X-\lambda B\overleftarrow{\delta}Z) \tag{2}$$

AF1 is of type I whereas AF2 and AF3 are of type II. For present purposes the factors may be interchanged, the roles of X and Z interchanged, or the splitting of a second difference into two first differences performed the reverse way without changing any of the following analysis.

The objective of the rest of this paper is to determine what values need to be placed on the various parameters in the problem in order to obtain rapid convergence. The parameters in question are λ (and possibly A_1), σ, and a range of values for α.

Von Neumann Stability Analysis

Following Ref. 7 let $e^n=\phi^n-\phi$ be the error after the nth iteration. Then $\Delta=e^{n+1}-e^n$ and

$$N(e^{n+1}-e^n)=\alpha\sigma L(e^n) \tag{3}$$

assuming for the moment that A and B have been evaluated using the converged solution ϕ [so that $L(\phi)=0$]. The Von Neumann analysis consists of writing

$$e^n(X,Z)=\sum_{p,q=1}^{\infty} G^n(p,q)\exp(ipX)\exp(iqZ) \tag{4}$$

substituting this into Eq. (3) and considering only a single Fourier component. (Strictly speaking, the analysis is valid only for constant A and B and for periodic boundary conditions, but in practice it often yields useful results with other types of boundary conditions.)

Following this procedure but ignoring the effect of the operators in the first factor on A_1, λ, and B (i.e., assuming them to be constant over distances of the order of a step length), the expressions below are obtained for the three types of schemes:

$$\text{I:}\quad (G^{n+1}-G^n)[\alpha^2/\lambda+4\alpha(AP+BQ)+16AB\lambda PQ]=-4\alpha\sigma(AP+BQ)G^n$$

$$\text{II:}\quad (G^{n+1}-G^n)[\alpha^2/\lambda\cdot 2(P+i\bar{P})+4\alpha(AP+BQ)+8AB\lambda(P-i\bar{P})Q]=-4\alpha\sigma(AP+BQ)G^n$$

$$\text{III:}\quad (G^{n+1}-G^n)[\alpha^2/\lambda\cdot 4(P+i\bar{P})(-Q+i\bar{Q})+4\alpha(AP+BQ)+4AB\lambda(-P+i\bar{P})(Q+i\bar{Q})]=-4\alpha\sigma(AP+BQ)G^n$$

where

$$P=\sin^2(\tfrac{1}{2}p\Delta X),\quad \bar{P}=\sin(\tfrac{1}{2}p\Delta X)\cos(\tfrac{1}{2}p\Delta X)$$

$$Q=\sin^2(\tfrac{1}{2}q\Delta Z),\quad \bar{Q}=\sin(\tfrac{1}{2}q\Delta Z)\cos(\tfrac{1}{2}q\Delta Z) \tag{5}$$

The amplification factor β ($=G^{n+1}/G^n$) for each of the types is given by

$$\text{I:}\quad \beta=\frac{\alpha^2+4\alpha\lambda(AP+BQ)(1-\sigma)+16\lambda^2ABPQ}{\alpha^2+4\alpha\lambda(AP+BQ)+16\lambda^2ABPQ} \tag{6}$$

$$\text{II:}\quad |\beta|^2=\frac{[\alpha^2P+2\alpha\lambda(AP+BQ)(1-\sigma)+4\lambda^2ABPQ]^2+[\alpha^2-4\lambda^2ABQ]^2P(1-P)}{[\alpha^2P+2\alpha\lambda(AP+BQ)+4\lambda^2ABPQ]^2+[\alpha^2-4\lambda^2ABQ]^2P(1-P)} \tag{7}$$

$$\text{III:}\quad |\beta|^2=\frac{[-(\alpha^2+\lambda^2AB)c+\alpha\lambda(AP+BQ)(1-\sigma)]^2+[\alpha^2-\lambda^2AB]^2s^2}{[-(\alpha^2+\lambda^2AB)c+\alpha\lambda(AP+BQ)]^2+[\alpha^2-\lambda^2AB]^2s^2}$$

with

$$c = \sin(\tfrac{1}{2}p\Delta X)\sin(\tfrac{1}{2}q\Delta Z)\cos\tfrac{1}{2}(p\Delta X - q\Delta Z)$$

$$s = \sin(\tfrac{1}{2}p\Delta X)\sin(\tfrac{1}{2}q\Delta Z)\sin\tfrac{1}{2}(p\Delta X - q\Delta Z)$$

For stability, and hence convergence, we must have $|\beta| < 1$ for all values of p and q. This implies:

$$\text{I:} \quad 0 < \sigma < 2 + \frac{\alpha^2 + 16\lambda^2 ABPQ}{2\alpha\lambda(AP+BQ)}$$

$$\text{II:} \quad 0 < \sigma < 2 + \frac{P(\alpha^2 + 4\lambda^2 ABQ)}{\alpha\lambda(AP+BQ)}$$

$$\text{III:} \quad 0 < \sigma < 2 - \frac{2(\alpha^2 + \lambda^2 AB)c}{\alpha\lambda(AP+BQ)}$$

For positive α and λ, I and II should converge for $0 < \sigma \le 2$. It can be shown, however, for the special case $A = B$, that $|\beta|$ cannot be kept below unity for all p and q in case III, since with $P = Q = 1$ the above condition on σ is

$$0 < \sigma < -\frac{2(\alpha - \lambda A)(\alpha - \lambda B)}{\alpha\lambda(A+B)}$$

so that wherever $A = B$ this upper limit becomes negative and there are no values of σ that will give $|\beta| < 1$ for all frequencies. Hence III is an unstable factorization and will not be considered further.

Choice of λ and α that Minimize Amplification Factor

For rapid convergence $|\beta|$ should be made as small as possible. We look for stationary values of $|\beta|$ with respect to λ with a view to choosing λ in order to minimize $|\beta|$. It will be found that values for α will emerge at the same time that minimize $|\beta|$ for particular frequencies.

In scheme I, $|\beta|$ will be zero if the numerator of Eq. (6) is zero, which can happen, for real α and λ, only if $\sigma > \sigma_I$, where $\sigma_I = 1 + 2\sqrt{ABPQ}/(AP+BQ)$. Since $\sigma_I < 2$, then if we choose $\sigma = 2$ the numerator of Eq. (6) will be zero if $\lambda = \alpha/4AP$ or $\lambda = \alpha/4BQ$. If we restrict dependence on the particular frequency to α while using λ to determine the best disposition for A and B in the factorization (2), then a factorization, called here Ia, that is in some sense optimum is characterized by:

$$\text{Ia:} \quad \sigma = 2, \quad \lambda = 1/B, \quad \alpha = 4Q \tag{8}$$

The other root, $\lambda = 1/A$, $\alpha = 4P$, gives a similar scheme with the roles of X and Z reversed.

If $\sigma < \sigma_I$ then $|\beta|$ has a minimum when $16\lambda^2 ABPQ = \alpha^2$, leading to a factorization, called here Ib, characterized by:

$$\text{Ib:} \quad \lambda = 1/\sqrt{AB}, \qquad \alpha = 4\sqrt{PQ} \tag{9}$$

However, if the choice of λ and α in Eq. (9) is made and σ exceeds σ_I, this leads to a local maximum in $|\beta|$ rather than to a minimum. Nevertheless it may be preferable to allow values of σ that will sometimes lead to $\sigma > \sigma_I$ for some frequencies and for regions of the flow where A and B take values making σ_I close to 1, rather than to restrict σ to the minimum value of σ_I (i.e., unity).

Turning now to scheme II, it can be shown, after some algebra, that $|\beta|$ is minimized with respect to λ (or α) if $\alpha^2 - 4\lambda^2 ABQ = 0$, leading to an optimum factorization characterized by:

$$\text{II:} \quad \lambda = 1/\sqrt{AB}, \quad \alpha = 2\sqrt{Q} \tag{10}$$

Choice of σ

For schemes Ib and II we still need to choose an optimum value for σ. For scheme Ib, if we substitute the values for λ and α from Eq. (9) into the expression for β in Eq. (6) we find

$$|\beta|_{\min} = \frac{2\sqrt{ABPQ} + (1-\sigma)(AP+BQ)}{2\sqrt{ABPQ} + AP + BQ}$$

for the particular frequencies corresponding to the value of α chosen. $|\beta|_{\min}$ is bounded above by the maximum of $\sigma - 1$ (when $ABPQ = 0$) and $1 - \tfrac{1}{2}\sigma$ (when $AP = BQ$). Thus the maximum value of $|\beta|_{\min}$ is minimized by choosing the value of σ that satisfies $\sigma - 1 = 1 - \tfrac{1}{2}\sigma$, that is, $\sigma = 4/3$, when the maximum value of $\beta_{\min}$ is $\tfrac{1}{3}$.

A similar argument holds for scheme II, where

$$|\beta|_{\min} = \frac{2P\sqrt{ABQ} + (1-\sigma)(AP+BQ)}{2P\sqrt{ABQ} + AP + BQ}$$

when λ and α from Eq. (10) are substituted into Eq. (7). $|\beta|_{\min}$ is again bounded above by the maximum of $\sigma - 1$ (when $ABPQ = 0$) and $1 - \tfrac{1}{2}\sigma$ (when $P = 1$, $A = QB$). Thus $|\beta|_{\min}$ is minimized by choosing $\sigma = 4/3$.

Optimum Factorizations

By the above analysis we have found three optimum factorizations:

$$\text{Ia:} \quad N = A_I[\alpha - (A/B)\overleftarrow{\delta X}\overrightarrow{\delta X}](B/A_I)(\alpha - \overleftarrow{\delta Z}\overrightarrow{\delta Z})$$

$$\text{Ib:} \quad N = A_I[\alpha - \sqrt{A/B}\overleftarrow{\delta X}\overrightarrow{\delta X}](1/A_I)(\alpha\sqrt{AB} - B\overleftarrow{\delta Z}\overrightarrow{\delta Z})$$

$$\text{II:} \quad N = A_I[\alpha - \sqrt{A/B}\overleftarrow{\delta X}](1/A_I)(\alpha\sqrt{AB}\overrightarrow{\delta X} - B\overleftarrow{\delta Z}\overrightarrow{\delta Z})$$

A convenient choice for A_I might be B for scheme Ia and $\sqrt{B}$ for schemes Ib and II, so that the three schemes become:

$$\text{Ia:} \quad N = (\alpha B - A\overleftarrow{\delta X}\overrightarrow{\delta X})(\alpha - \overleftarrow{\delta Z}\overrightarrow{\delta Z}) \tag{11a}$$

with $\sigma = 2$ and α chosen from

$$\alpha = 4\sin^2(\tfrac{1}{2}q\Delta Z) \tag{11b}$$

$$\text{Ib:} \quad N = (\alpha\sqrt{B} - \sqrt{A}\overleftarrow{\delta X}\overrightarrow{\delta X})(\alpha\sqrt{A} - \sqrt{B}\overleftarrow{\delta Z}\overrightarrow{\delta Z}) \tag{12a}$$

with $\sigma = 1\tfrac{1}{3}$ and α chosen from

$$\alpha = 4\,|\sin(\tfrac{1}{2}p\Delta X)\cos(\tfrac{1}{2}q\Delta Z)| \tag{12b}$$

$$\text{II:} \quad N = (\alpha\sqrt{B} - \sqrt{A}\overleftarrow{\delta X})(\alpha\sqrt{A}\overrightarrow{\delta X} - \sqrt{B}\overleftarrow{\delta Z}\overrightarrow{\delta Z}) \tag{13a}$$

with $\sigma = 1\tfrac{1}{3}$ and α chosen from

$$\alpha = 2\,|\sin(\tfrac{1}{2}q\Delta Z)| \tag{13b}$$

Scheme Ia would appear to be a simple scheme to apply, especially since A and B occur only in the first factor and thus do not need recalculating when solving for the second factor. However, there are practical objections to schemes Ib and II:

1) A and B need evaluating during the second as well as the first sweep, thus increasing the computational effort.

2) Continually taking square roots is costly in computer time.

3) When the flow is locally supersonic it would be difficult to model the upwind differencing in N.

4) The operators in the first bracket will act on the $\sqrt{A}$ and $\sqrt{B}$ in the second bracket with unpredictable effect; this effect was ignored in the analysis.

A Practical Variation of Schemes Ib and II

For most problems A and B are products of a function of the flow variables (e.g., a^2-u^2) and transformation derivatives arising from the mapping of the physical to the computational plane, so that it is practical to write

$$A=\bar{A}H_A(X,Z), \quad B=\bar{B}H_B(X,Z)$$

where H_A and H_B are independent of ϕ, and far from the solid boundaries (e.g., airfoil surface)

$$\bar{A}\to\bar{A}_\infty, \quad \bar{B}\to\bar{B}_\infty$$

An alternative to Eq. (9), which arose from satisfying $16\lambda^2ABPQ=\alpha^2$, is

$$\alpha=4\sqrt{\bar{A}_\infty/\bar{B}_\infty}\sqrt{PQ}, \quad \lambda=\sqrt{\bar{A}_\infty/\bar{B}_\infty}1/B\sqrt{B/A}\sim 1/B\sqrt{H_B/H_A}$$

where we have extracted the factor B so that it will be absent from the second factor (see below) and then included only the variation in transformation derivatives in λ. Scheme Ib now becomes, with $A_1=B$,

$$\text{Ib:}\quad N=(\alpha B-A\sqrt{H_B/H_A}\bar{\delta}X\bar{\delta}X)(\alpha\sqrt{H_A/H_B}-\bar{\delta}Z\bar{\delta}Z)$$

with $\sigma=1\frac{1}{3}$ and α chosen from

$$\alpha=4\sqrt{\bar{A}_\infty/\bar{B}_\infty}\,|\sin(\tfrac{1}{2}p\Delta X)\sin(\tfrac{1}{2}q\Delta Z)|$$

In a similar fashion scheme II becomes

$$\text{II:}\quad N=(\alpha B-A\sqrt{H_B/H_A}\bar{\delta}X)(\alpha\sqrt{H_A/H_B}\bar{\delta}X-\bar{\delta}Z\bar{\delta}Z) \quad (14a)$$

with $\sigma=1\frac{1}{3}$ and α chosen from

$$\alpha=2\sqrt{\bar{A}_\infty/\bar{B}_\infty}\,|\sin(\tfrac{1}{2}q\Delta Z)| \quad (14b)$$

The application of this scheme to a practical example will be demonstrated later.

Choice of Sequence of Values for α

Return now to scheme II as expressed in Eqs. (13). For each value of α chosen, the error for all frequencies will be reduced during the iteration (since $|\beta|<1$ for all frequencies) but will be reduced by the largest amount for frequencies close to $(2/\Delta Z)\sin^{-1}(\alpha/2)$ [from Eq. (13b)]. If we choose a range of values of α, say, $\alpha_K=2\sqrt{Q_K}$ $(K=1,\ldots,N)$, then with $\lambda=1/\sqrt{AB}$, the amplification factor for each K and for frequencies corresponding to P and Q $(p=2/\Delta X\sin^{-1}\sqrt{P},\ q=2/\Delta Z\sin^{-1}\sqrt{Q})$ is

$$\begin{aligned}&[(\sqrt{AB}P(Q_K+Q)+\sqrt{Q_K}(AP+BQ)(1-\sigma))^2\\&+AB(Q_K-Q)^2P(1-P)]^{\frac{1}{2}}\\&\div[(\sqrt{AB}P(Q_K+Q)+\sqrt{Q_K}(AP+BQ))^2\\&+AB(Q_K-Q)^2P(1-P)]^{\frac{1}{2}}\end{aligned}$$

With some algebra it may be shown that this takes its maximum value where $AP=BQ$. Denoting this maximum by $\beta_K(P,Q)$, we find

$$\beta_K(P,Q)=\frac{\{[\sqrt{P}(Q_K+Q)+2\sqrt{QQ_K}(1-\sigma)]^2+(Q_K-Q)^2(1-P)\}^{\frac{1}{2}}}{Q_K+Q+2\sqrt{PQQ_K}}$$

Thus after N iterations, with α taking a range of values, the maximum error for the particular frequencies p, q will be reduced by a factor

$$\prod_{K=1}^{N}\beta_K$$

The mean reduction per iteration will thus be

$$\bar{\beta}=\left(\prod_{K=1}^{N}\beta_K\right)^{1/N}$$

and our objective will be to find a sequence of β_Ks and a value of N that minimizes the maximum value of $\bar{\beta}$, since, for convergence, we need to decrease the errors corresponding to all frequencies successively and the speed of convergence will depend on how small is the largest amplification factor. Each β_K has its minimum where $Q=Q_K$ and grows with $|Q-Q_K|$. If Q lies between Q_K and Q_{K+1} then the major reduction to the error for the frequency will come from β_K and β_{K+1}. Thus we should expect that the amplification factor will be largest if Q is such that $\beta_K=\beta_{K+1}$, that is, $Q=\sqrt{Q_KQ_{K+1}}$, when

$$\begin{aligned}\beta_K=\beta_{K+1}=\{&[\sqrt{P}(1+\sqrt{Q_{K+1}/Q_K})+(Q_{K+1}/Q_K)^{\frac{1}{4}}(1-\sigma)]^2\\&+(\sqrt{Q_{K+1}/Q_K}-1)^2(1-P)\}^{\frac{1}{2}}/[1+\sqrt{Q_{K+1}/Q_K}\\&+2\sqrt{P}(Q_{K+1}/Q_K)^{\frac{1}{4}}]\end{aligned}$$

Now, we wish to minimize the maximum value of $\bar{\beta}$, and, since $\bar{\beta}$ will have several maxima, the upper bound of $\bar{\beta}$ should be least if all the maxima are equal. This leads to $Q_{K+1}/Q_K=\text{const}=a^2$, say, or $Q_K=Q_1a^{2(K-1)}$ $(K=1,2,\ldots,N)$.†

The lowest and highest values of Q are $\frac{1}{4}\Delta Z^2$ $(q=1)$ and $1(q\Delta Z=\pi)$, respectively. Thus, putting $Q_1=\frac{1}{4}\Delta Z^2$ and $Q_N=1$, we have the sequence

$$Q_K=Q_1a^{2(K-1)}, \quad (K=1,2,\ldots,N)$$

with

$$Q_1=\tfrac{1}{4}\Delta Z^2 \text{ and } a=(2/\Delta Z)^{1/(N-1)} \quad (15)$$

or

$$\alpha_K=\alpha_1a^{K-1} \text{ with } \alpha_1=\Delta Z, \quad a=(2/\Delta Z)^{1/(N-1)}$$

This is the sequence suggested by Ballhaus et al.,[7] and we see that there is some theoretical justification for it. In fact, the above argument really applies only for values of Q near the middle of the range (near $Q_{N/2}$), since contributions to $\bar{\beta}$ come from each β_K and near the ends of the range (say, near Q_1) the contribution from the other end (near Q_N) will be close to unity. It will be seen later [sequence (16) below] that this end effect can be counteracted to some extent by repeating the endpoints. With sequence (15) it is found in practice that $\bar{\beta}$ is larger for small P than for large P (it can be shown that β_K has its maximum value with respect to P at $P=0$), so that we need to find the value of N that minimizes the maximum value of $\bar{\beta}$ when P takes its smallest value $(\frac{1}{4}\Delta X^2)$. As an example, take $\sigma=1\frac{1}{3}$, $2/\Delta X=64$, and $2/\Delta Z=32$ (a 64×32 grid if X and Z extend from -1 to $+1$). Figure 1 (lower) shows $\bar{\beta}$ plotted against Q (Q on a logarithmic scale) for $N=4$, 5, and 8. β_{max} for the three cases is 0.745, 0.732, and 0.758. For this case $N=5$ should give fastest convergence. A similar calculation for a 32×16 grid and a 128×64 grid, with this sequence, produces best values of N of 4 and 6, respectively. Comparing

†It has been brought to the author's attention that a similar argument for a simplified version of scheme Ia, reaching almost the same conclusions, is contained in Ref. 11.

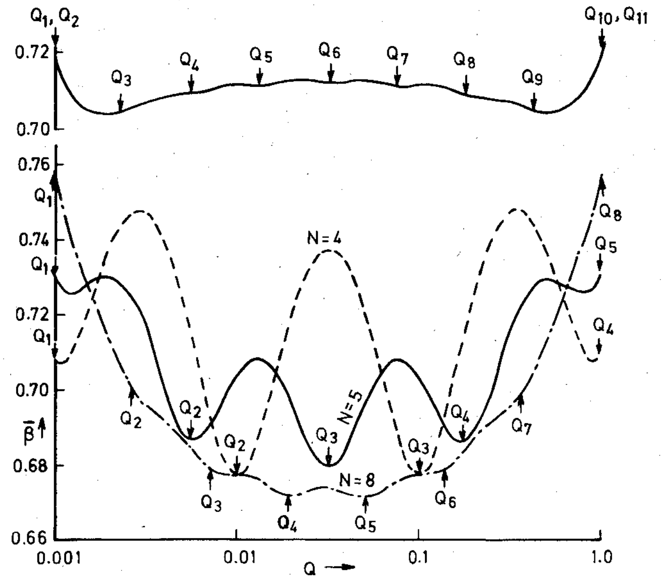

Fig. 1 Mean amplification factor with geometric sequence [Eq. (15)] for Q_K (lower) and with repeated endpoints [Eq. (16)] (upper).

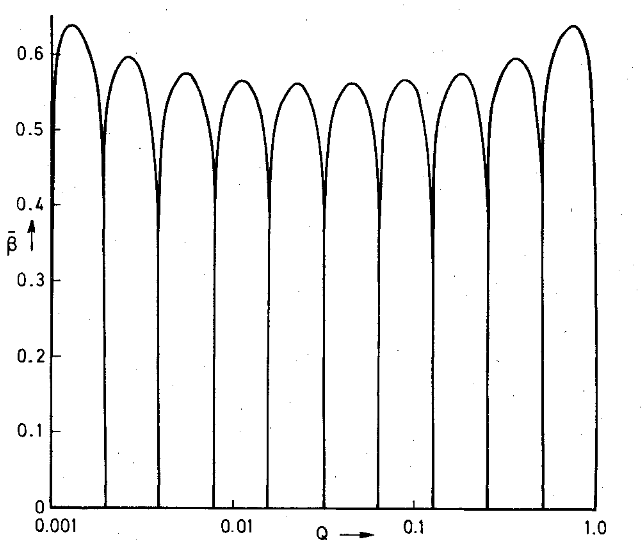

Fig. 2 Mean amplification factor with geometric sequence [Eq. (15)] for Q_K, scheme Ia.

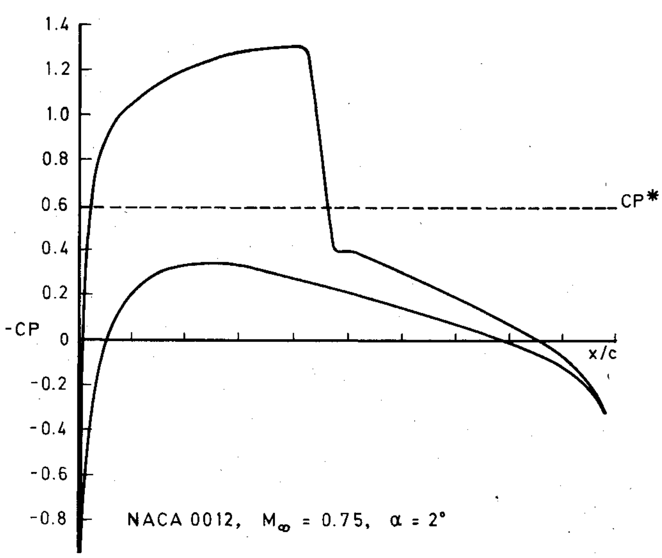

Fig. 3 Nonconservative potential flow solution for NACA 0012 airfoil: $M_\infty = 0.75$; incidence = 2 deg; $C_L = 0.474$.

the $N=4$ and $N=5$ curves in Fig. 1 (lower), we see that although the $N=4$ curve has smaller minima it also has larger maxima. It is the latter that are the more important, since, for convergence, the errors for *all* frequencies must be damped out. It would seem that the ideal situation would be to have $\bar{\beta}$ constant for all Q. Large values of N produce a much smoother curve (e.g., the curve for $N=8$) but unfortunately give large values of $\bar{\beta}$ at the ends (Q_1 and Q_N). An improvement is possible by repeating Q_1 and Q_N, giving the sequence

$$Q_1 = \tfrac{1}{4}\Delta Z^2, \quad Q_K = Q_1 a^{2(K-2)} \quad (K=2,\ldots,N-1),$$
$$Q_N = 1 \quad \text{with} \quad a = (2/\Delta Z)^{1/(N-3)} \tag{16}$$

For the three abovementioned grids, 32×16, 64×32, and 128×64, best values of N are found to be 9, 11, and 12, respectively. $\bar{\beta}$ is plotted against Q for this sequence in Fig. 1 (upper) for the 64×32 grid (again $\sigma = 1\tfrac{1}{3}$ and $P = \tfrac{1}{4}\Delta X^2$), and it can be seen that $\bar{\beta}$ is much closer to being constant, with $\bar{\beta}_{max} = 0.720$. We thus suggest the sequence

$$\alpha_1 = \Delta Z, \quad \alpha_K = \alpha_1 (2/\Delta Z)^{(K-2)/(N-3)}$$
$$(K=2,3\ldots,N-1), \quad \alpha_N = 2 \tag{17}$$

as being near optimum. With some effort, doubtless a better sequence of αs could be found, but the above sequence has the advantage of simplicity.

For schemes Ia and Ib, corresponding expressions for β_K are:

$$\text{Ia:} \quad \beta_K(Q) = \frac{|Q-Q_K|}{Q+Q_K} \quad \left(\text{when } \frac{AP}{BQ} = 0 \text{ or } \infty\right)$$

$$\text{Ib:} \quad \beta_K(R) = \frac{R_K^2 + R^2 + 2R_K R(1-\sigma)}{R_K^2 + R^2 + 2R_K R} \quad (\text{when } AP = BQ)$$

where $R = PQ$.

The curves of $\bar{\beta}$ against R for scheme Ib look quite similar to those in Fig. 1, but for scheme Ia the geometric sequence [Eq. (15)] results in the behavior shown in Fig. 2, which is for $2/\Delta Z = 32$ and $N=11$. Thus, even though $\bar{\beta}$ is zero for certain frequencies ($Q = Q_K$), for scheme Ia this is of little advantage since it is the maximum value of $\bar{\beta}$ rather than its minimum value that determines the convergence rate. Sequences of α similar to that for scheme II [sequence (17)] appear to be near optimum:

$$\text{Ia:} \quad \alpha_1 = \Delta Z^2, \quad \alpha_K = \alpha_1 (4/\Delta Z^2)^{(K-2)/(N-3)}$$
$$(K=2,\ldots,N-1), \quad \alpha_N = 4 \tag{18}$$

$$\text{Ib:} \quad \alpha_1 = \Delta X \Delta Z, \quad \alpha_K = \alpha_1 (4/\Delta X \Delta Z)^{(K-2)/(N-3)}$$
$$(K=2,\ldots,N-1), \quad \alpha_N = 4 \tag{19}$$

Optimum values of N and corresponding maximum values of $\bar{\beta}$ for the three schemes, when using the sequences (17, 18, and 19), are given in Table 1.

Example: Stretched Cartesian Grid

The example chosen is the popular one of transonic flow around a NACA 0012 airfoil at a Mach number of 0.75 and an incidence of 2 deg. The pressure distribution resulting from a nonconservative computation using the full-potential equation with a stretched Cartesian coordinate system (airfoil boundary conditions being satisfied by interpolation) is shown in Fig. 3.

Scheme II ought to be more suitable for transonic-flow problems[7] and in any case has been found to be considerably

Table 1 Optimum values of N and maximum mean amplification factors

Grid size:	32×16		64×32		128×64	
	N	$\bar{\beta}$	N	$\bar{\beta}$	N	$\bar{\beta}$
Ia	15	0.538	17	0.596	20	0.641
Ib	8	0.683	10	0.717	11	0.743
II	9	0.663	11	0.720	12	0.754

faster than either of the other schemes. Most of the test runs have thus been made using scheme II. For the stretched Cartesian grid we have

$$A=(a^2-u^2)(X_x)^2/\Delta X^2 \qquad B=(a^2-w^2)(Z_z)^2/\Delta Z^2$$

where u and w are velocity components in the x and z directions, a is the speed of sound, and X and Z are coordinates in the computational plane $(-1\leq X,\ Z\leq +1)$. If the freestream is in the x direction, then far from the airfoil

$$A\sim(a_\infty^2-1)(X_x)^2/\Delta X^2 \qquad B\sim a_\infty^2(Z_z)^2/\Delta Z^2$$

and a practical variation of scheme II is

$$\left(\alpha B-A\frac{Z_z}{X_x}\frac{\Delta X}{\Delta Z}\overleftarrow{\delta X}\right)\left(\alpha\frac{X_x}{Z_z}\frac{\Delta Z}{\Delta X}\overrightarrow{\delta X}-\overleftarrow{\delta Z}\overrightarrow{\delta Z}\right)\Delta=\alpha\sigma L(\phi) \qquad (20)$$

where $L(\phi)$ is now the discretized form of the full-potential equation. The first factor needs to be modified in regions of supersonic flow. If rotated differences are applied, then A and B will be formed partly by central differencing and partly by upwind differencing.[9] If the two parts are denoted by A_c, B_c and A_u, B_u, then in supersonic regions the first factor is replaced by

$$\alpha(B_c+B_u)-\frac{Z_z}{X_x}\frac{\Delta X}{\Delta Z}(A_c\overleftarrow{\delta X}+A_u\overleftarrow{\overleftarrow{\delta X}})$$

We try first a computation on a 64×32 grid ($\Delta X=1/32$, $\Delta Z=1/16$) with $\sigma=1\frac{1}{3}$ and the following sequence for α:

$$\alpha_I=2\sqrt{1-M_\infty^2}, \quad \alpha_K=\alpha_I(\alpha_N/\alpha_I)^{(K-2)/(N-3)}$$
$$(K=2,3,\ldots,N-1), \quad \alpha_N=\Delta Z\sqrt{1-M_\infty^2} \qquad (21)$$

and $N=11$. This is the same sequence (except for the factor $\sqrt{1-M_\infty^2}$) as in Eq. (17), but in the reverse order, high values of α coming first, since this order reversal has been found to improve stability near the start of the computation. Figure 4 shows the convergence history. The residual is the value of $L(\phi)$ in Eq. (20), normalized with respect to its value on the first iteration and plotted on a logarithmic scale. For practical purposes the solution can be considered to be converged after 20-30 iterations.

Variations from the Optimum

To illustrate that the preceding analysis has in fact led to a near-optimum scheme, the effect of departing from the scheme will be investigated. In particular, we will vary in turn the contents of the two factors, the value of the relaxation factor σ, the maximum and minimum values of α, and the number N of terms in the sequence.

The importance of correctly splitting the transformation derivatives between the factors is illustrated by replacing the factorization (20) by the simpler one

$$(\alpha B-A\overleftarrow{\delta X})(\alpha\overrightarrow{\delta X}-\overleftarrow{\delta Z}\overrightarrow{\delta Z})\Delta=\alpha\sigma L(\phi)$$

Table 2 Number of iterations needed to reduce maximum residual by a factor of 10^5

	σ	N	$\alpha_I/\alpha_I(\text{opt})$	$\alpha_N/\alpha_N(\text{opt})$	Number of iterations
Optimum→	$1\frac{1}{3}$	11	1	1	50
	1.1				81
	1.2				71
	1.3				69
	1.4				57
	1.5				60
		7			86
		8			75
		9			67
		10			73
		12			51
		13			54
		14			52
			0.5		79
			2.0		53
				0.5	58
				2.0	70

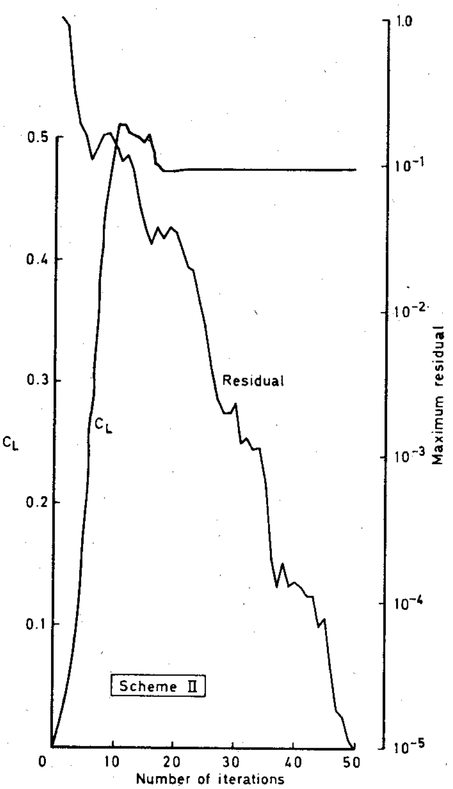

Fig. 4 Convergence history for optimum scheme.

suitably modified in supersonic regions and with

$$\alpha_K=\alpha_I(\alpha_N/\alpha_I)^{(K-1)/(N-1)} \qquad (K=1,\ldots,N)$$

Figure 5 shows the convergence history when $\sigma=1\frac{1}{3}$, $N=10$, $\alpha_I=1$, and $\alpha_N=1/8$. (Various other combinations were tried but none of them produced better results.) Note that the horizontal scale has been stretched fivefold compared with Fig. 4 and the solution is still far from converged after 300 iterations.

Next each of the parameters σ, N, α_I, and α_N was varied in turn. Table 2 gives the number of iterations needed to reduce the maximum residual by a factor of 10^5.

The results of Table 2 show remarkable qualitative agreement with the analysis. Table 1 suggests that a mean

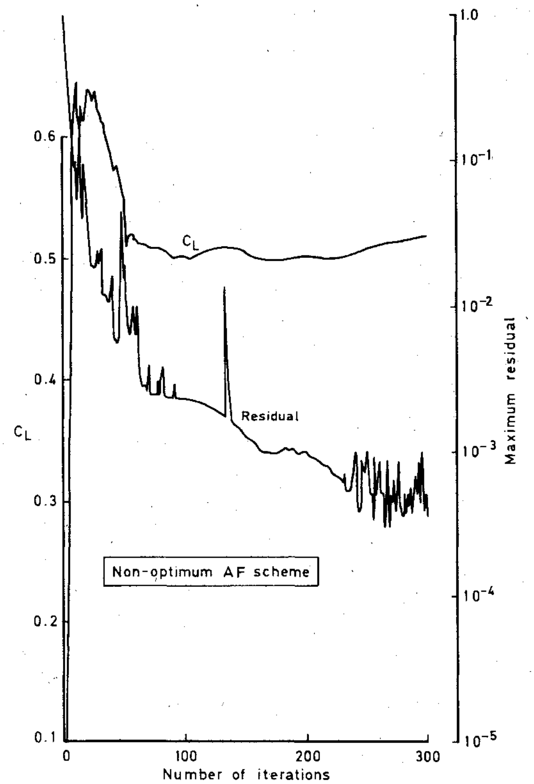

Fig. 5 Convergence history for nonoptimum scheme.

reduction of 1/0.72 per iteration can be expected. This would lead to reduction by a factor of 10^5 in 35 iterations, whereas in practice 50 are needed. Evidently convergence is delayed because of the growing importance of other terms in the differential equation as the airfoil is approached (and also possibly because of the airfoil boundary condition), as well as the appearance of a supersonic region in the flow (excluded from the analysis) and the use of a scheme that is only an approximation to the theoretical optimum.

The other two schemes, Ia and Ib, were tried for the same example, but convergence was far slower, scheme Ia being about one-fifth the speed of scheme II and scheme Ib being even slower. There was also much poorer agreement between optimum values for the various parameters established from the theory and found in practice, although the theoretical ones did provide a useful initial set of values that could be varied in turn. According to Table 1 the Ia and Ib schemes should be faster than scheme II. The author cannot explain why they are so poor (a coding error could, of course, be responsible), unless the other terms in the differential equation that become important near the airfoil (and the other reasons mentioned at the end of the previous paragraph) have a relatively larger effect in delaying convergence. Scheme Ia was no better when applied to a subcritical case.‡

Concluding Remarks

The above example has shown, at least for factorizations where differencing in one of the directions is split between the two factors, that an examination of the stability and convergence properties of the scheme, applied far enough away from the airfoil for lower-order terms in the difference equation to be negligible, can give an excellent guide to how to choose the factorizations, relaxation parameter, and acceleration parameters. The major result is the observation that it is important to correctly split the transformation derivatives between the two factors. The analysis applies equally if the roles of X and Z are switched, sweep directions are reversed, or the factors are reversed, so that in particular it will apply to factorizations of the AF3 type,[9] which have so far been the most successful. The choice of $\sigma = 1\frac{1}{3}$ appears to be confirmed as optimum by numerical experience, as also have the minimum and maximum values of the acceleration parameter α. Improvements can undoubtedly still be made to the choice of sequence of values of α, although it seems that an optimum sequence will probably not depart too much from the usual[7] geometric one. Even though the values of α from this sequence may be near-optimum ones to choose (remembering that, for each value of α chosen, errors of frequencies within a certain range are reduced the most), the order in which they are taken may be important. The author has not experimented in this area, except for discovering that large values of α should be used at the start of a calculation (assuming that uniform flow is taken as the initial guess to the flowfield) in order to avoid the solutions immediately diverging. It may be preferable to use the larger values of α [which reduce the high-frequency (or local) errors] in the latter stages of the computation in order to reduce the residual quickly once the major features of the flow, such as shock position and circulation, have stabilized.

It has not been found possible to extend this analysis to the three-dimensional potential-flow equation, except where it can justifiably be assumed that changes in one coordinate direction are small compared with those in the other direction, in which case a 2D analysis is applicable.

‡Note added in proof: The author now believes that the slow convergence derives from the boundary conditions that are applied to the intermediate variable F (see section above entitled "Approximate Factorization"), which is solved for during the first AF sweep. The airfoil boundary condition is of Neumann type, which appears to be more compatible with scheme II, whereas schemes Ia and Ib seem to be faster when a Dirichlet boundary condition is applied.

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